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Batyrev's Construction of Mirror pairs

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ABSTRACT: It's not Abstract

Contents

We have already know the basic concepts of Mirror symmetry, but not surprisingly, it's quite difficult to construct mirror pairs. One way to deal with this problem is by considering simpler kind of Calabi-Yau manifolds, namely the toric Calabi-Yau manifolds. As they are constructed algebraically, numerous methods can be applied to compute its mirror pairs explicitly. One of the most important technique on constructing toric Calabi-Yau's mirror manifold is Batyrev's construction.

1 Recalling Toric geometry and Mirror symmetry

From our previous knowledge of Toric geometry, we know that their are two lattices where basic elements lies. They are the lattice where the polytopes are defined and dual lattice where the fans are defined. We Shall first have a brief recall of these concepts.

1.1 Recall:toric variety

A lattice is a space isomorphism to $\mathbb{Z}^n$,let $N = Hom(M, \mathbb{Z})$ be another lattice which is denoted as its dual. A cone, or a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}} = N \otimes \mathbb{R}$

$$\sigma = \{\sum_{i=1}^{s} \lambda_i u_i : \lambda_i \geq 0, u_i \in N\} \tag{1.1}$$

we say σ is strongly convex if $\sigma \cap (-\sigma)$. A dual cone of every cone is defined by

$$\breve{\sigma} = \{m \in M_{\mathbb{R}} : \langle m, v\rangle \geq 0, \forall v \in \sigma\} \tag{1.2}$$

To define fans, we have defined face of a cone by

$$\tau = \{v \in \sigma : \langle m, v\rangle = 0\} \subseteq \sigma \tag{1.3}$$

and a fan is a collection of cones which every cone's faces is a element and the intersecting part of two elements is also a element, often we denote a fan as Σ. From the classical definition of toric variety, we first have a variety

$$X_\sigma = Spec\{\mathbb{C}[M \cap \breve{\sigma}]\} \tag{1.4}$$

where $\mathbb{C}[M \cap \breve{\sigma}]$ is the $\mathbb{C}$ algebra with generators $w^m = \Pi w_i^{m_i}$ for each $m \in M \cap \breve{\sigma}$.

Now we look at the original lattice, a polytope $\Delta \subset M_{\mathbb{R}}$ is defined as a convex hull of a finite set of points. Numerous ways can be used to get a bigger polytope from smaller ones, a simple way is from convex hull by $Conv(\Delta_1, ..., \Delta_k)$, another way is by Minkowski sum as

$$\Delta_1 + ... + \Delta_k = \{m_1 + ... + m_k : m_i \in \Delta_i\} \tag{1.5}$$

and $k\Delta = \Delta + ... + \Delta$ is often used.

Batyrve has a way to naturally construct a toric variety from a polytope (we use the noun polytope as polytope whose vertices are all integers, this is called integral polytope in some literatures): Given Δ , consider all monomials which has the form

$$w_0^k w^m \tag{1.6}$$

where $m \in k\Delta$, obviously there is a $\mathbb{C}$ algebra of graded k generated by these monomials, which is called the polytope ring S_Δ. This ring gives a projective variety as

$$\mathbb{P}_\Delta = Proj(S_\Delta) \tag{1.7}$$

Another understanding is directly from dual cones

$$\breve{\sigma}_F = \{\lambda(m - m') : m \in \Delta, m' \in F, \lambda \geq 0\} \subset M_{\mathbb{R}} \tag{1.8}$$

the dual of these dual cones together forms a fan which is regarded as the normal fan of Δ.Obviously there is a toric variety comes from here.

We claim that these two varieties are the same,by introducing the polar polytope

$$\Delta^\circ = \{v \in N_{\mathbb{R}} : \langle m, v\rangle \geq -1, \forall m \in \Delta\} \subset N_{\mathbb{R}} \tag{1.9}$$

If the origin is a interior point of the polytope, this polar polytope is well-defined and also contains the origin. The natural fan obtained by considering the cone of Δ° is the normal fan of the polytope Δ, and is equivalent to the projective variety.

Of course, the definition from $\mathbb{C}$ algebra is not explicit at all, so it is known that we often gets the toric variety directly by the 1-dim cone of the fan, a simple example for $\mathbb{P}^2$ is its 1-dim cone as

$$\{(0,1),(1,0),(-1,-1)\} \tag{1.10}$$

and the charges which satisfies $\sum_i Q_i u_i$ is $(1,1,1)$, which gives the equivalence relation $(...,w_i,...) \sim (...,\lambda^{Q_i} w_i,...)$, and singularities Z_Σ are defined by

$$Z_\Sigma = \bigcup_I \{(w_1,...,w_k) : w_i = 0 \forall i \in I\} \tag{1.11}$$

where $I \subseteq \{1,...,k\}$ for which $\{w_i : i \in I\}$ does not belong to a cone in Σ.And there we has the toric variety as

$$\frac{\mathbb{C}^n - Z_\Sigma}{\sim} \tag{1.12}$$

.We know this is equivalent to the $Spec\{\mathbb{C}[M \cap \check{\sigma}]$ definition,which is named as the homogeneous coordinate description,stricts the number of 1 dimensional cone should be

$$n_{\Sigma(1)} = rank(N_\mathbb{R}) + 1 \tag{1.13}$$

.If we have more or less then this value, we cannot find a global homogeneous description, for example we have

$$\Sigma(1) = \{v_1,..,v_p\} \tag{1.14}$$

and the dual lattice is in dimension $dim(N_\mathbb{R}) = r$,we can only construct the embedding

$$\phi(w_1,...,w_p) = (\prod_{i=1}^p w_i^{v_{i,1}},...,\prod_{i=1}^p w_i^{v_{i,r}}) \tag{1.15}$$

which also gives a r dimensional variety.

Also a similar construction is applied to polytopes,which describe the toric variety generated by polytope as a injective map of a higher dimensional projective space. For a polytope Δ , we denote all its intersection with the lattice as

$$\Delta \cap M = \{m_1,...,m_k\} \tag{1.16}$$

and we can generate k monomials by $w_i^{(m_i^j)}$, where $j \in \{1,...,k\}$. The similar variety is defined

$$(\mathbb{C}^*)^n \to \mathbb{P}^k : (w_1,...,w_n) \to [w_i^{(m_i^1)},...,w_i^{(m_i^n)}] \tag{1.17}$$

where $[...,...]$ is the homogeneous coordinate of the projective space.

We also should note when these two constructions of toric varieties gives Calabi-Yau spaces, which is ensured by the trivialness of canonical bundle or Calabi-Yau space has a nowhere vanishing n form. This gives the Calabi-Yau condition of toric variety that all the vector generators of the one dimensional cone lie in the same affine hyperplane, if we considers the global language, it means all charges sum up to 0

$$\sum_i Q_i = 0 \tag{1.18}$$

1.2 recall:Mirror symmetry

From our previous knowledge of 2d $N = (2,2)$ theory, we know mirror symmetry is a duality of two different models which has different chiralities, namely the A model and the B model. In the perspective of string compactification, superstrings generate SCFT theories on Calabi-Yau spaces where they are compactified, thus this duality is generalized to a duality of two Calabi-Yau spaces.

Calabi-Yau is a space which possess the Kähler structure and the Complex structure at the same time, and both these structures has a moduli space which is a Kähler space according to the deformation theory, and the dimension of their moduli space is $h^{1,1}$ and $h^{2,1}$ respectively(for Calabi-Yau threefold). Therefore when mirroring a Calabi-Yau threefold, we switch their moduli space and a simple result is

$$h^{1,1} = \tilde{h}^{2,1} \tag{1.19}$$

$$h^{2,1} = \tilde{h}^{1,1} \tag{1.20}$$

and it's not hard to see if we write all hodge numbers as a diamond, this flips the diamond in an axis, like a diamond in the mirror.

Although we focus on Calabi-Yau threefold in physics, it is worth noting that the relation above can be generalized to generic dimension of Calabi-Yau spaces

$$h^{1,1}(M) = h^{dim(M)-1,1}(\tilde{M}) \tag{1.21}$$

$$h^{dim(M)-1,1}(M) = h^{1,1}(\tilde{M}) \tag{1.22}$$

1.3 Example of construct toric varieties by polytopes

A simple example is the $\mathbb{P}^1 \times \mathbb{P}^1$(we should be alert that this is not the resolved conifold$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$,though they are the same when we draw their diagram,but the latter one is the diagram on a hypersurface,this one is the full diagram).

We consider the polytope $[0,1] \times [0,1]$ and its normal fan is composed by

$$\{(1,0),(0,1),(-1,0),(0,-1)\} \tag{1.23}$$

which is the space that glues these four varieties

$$Spec[\mathbb{C}(X,Y)], Spec[\mathbb{C}(X,Y^{-1})], Spec[\mathbb{C}(X^{-1},Y)], Spec[\mathbb{C}(X^{-1},Y^{-1})] \tag{1.24}$$

which can be viewed as two $\mathbb{P}^1$ space's coordinate. And if we associate all integer points to a monomial

$$\{(0,0),(0,1),(1,0),(1,1)\} \to \{1,y,x,xy\} \tag{1.25}$$

and we get a ring

$$\mathbb{C}[1,x,y,xy] \tag{1.26}$$

which is

$$\frac{C[w_1,w_2,w_3,w_4]}{w_1w_4 - w_2w_3} \tag{1.27}$$

this ring gives a projective variety that equals to $\mathbb{P}^1 \times \mathbb{P}^1$

$$Proj(\frac{C[w_1, w_2, w_3, w_4]}{w_1 w_4 - w_2 w_3}) = \mathbb{P}^1 \times \mathbb{P}^1 \tag{1.28}$$

which can be explicitly showed as the Segre embedding

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 : ([s : t], [u : v]) \to [su : sv : tu : tv] \tag{1.29}$$

1.4 remark on construct toric variety by polytope

From the example above, a quite confusing thing may occurs to our mind: does the construction of toric varieties from polytopes is well-defined? This is questioned since infinitely many polytopes gives the same normal fan, but does they all gives the same construction as the fan gives?

Recall the definition of normal fan of a polytope(we always discuss this for a integral polytope),

$$\sigma_F = \{v \in N_{\mathbb{R}} | \langle m, v \rangle \leq \left\langle m', v \right\rangle, \forall m \in F, m' \in \Delta\} \tag{1.30}$$

We shall expand our previous example to discuss this thing.

We examine the polytope

$$[0, 2] \times [0, 2] \tag{1.31}$$

now,this gives exactly the same normal fan that $[0, 1] \times [0, 1]$ gives.Therefore it gives exactly the same toric variety in the aspect of constructing from normal fans, but if we consider to construct a toric variety directly from polytope by consider its all integer points

$$\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\} \tag{1.32}$$

this gives us a algebra

$$\mathbb{C}[1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2] \tag{1.33}$$

and the toric variety is by this algebra's projective variety.

$$Proj(\mathbb{C}[1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2]) \tag{1.34}$$

we can understand this by a embedding called 2-Veronese embedding, as a embedding to $\mathbb{P}^8$

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 : ([s : t], [u : v]) \to [s^2u^2 : s^2uv : s^2v^2 : stu^2 : stuv : stv^2 : t^2u^2 : t^2uv : t^2v^2] \tag{1.35}$$

we can see this is what we want by taking

$$[s : t] = [1 : x], [u : v] = [1 : y] \tag{1.36}$$

this tells us not only the polytope gives a toric variety, but also gives a line bundle of how the toric variety is embedded to a higher dimensional projective space.Due to the connection between divisors and line bundle,

$$\mathcal{O}(D)(U) = \{f \in \mathcal{K} | div(f) + D \geq 0 \quad on \quad U\} \tag{1.37}$$

where $\mathcal{K}$ is all rational functions on U and $div(f)$ is the divisor of a rational function(or all it's zeros subtract its poles),this means the construction by polytope gives a toric divisor, which is a Ample toric divisor, the term Ample means there is a embedding to projective space by the line bundle it generates for some positive number times the divisor.

When a toric variety $\mathbb{P}_\Delta$ is constructed out of a polytope, it is automatically equipped with an ample toric divisor D defined as the pullback of the hyperplane class on projective space.Thus, the polytope construction yields strictly more data than the fan method.However, given a toric variety X_Σ together with an ample toric divisor, it is possible to reconstruct the polytope Δ that yields this pair.

Since

$$\{f \in \mathcal{K} | div(f) + D \geq 0 \quad on \quad U\} \simeq \Gamma(X_\Sigma, \mathcal{O}(D)) \tag{1.38}$$

is given by

$$f \to f\dot{s}_0 \tag{1.39}$$

where s_0 is given by the global meromorphic function for which $div(s_0) = D$,using this correspondence, the coordinate function x_i yield meromorphic functions f_i on X_Σ,which can be viewed as an element $m_i \in M$, and the polytope is thus the convex hull of $m_0, ..., m_k$. A simple example is we begin with the toric variety $\mathbb{P}^2$ and the toric divisor $D_0 = \{x_0 = 0\}$. Then $s_0 = x_0$ is a global meromorphic section whose divisor is D_0, so the functions f_i satisfies the relation has

$$f_i \dot{x}_0 = x_i \tag{1.40}$$

means $f_i = x_i/x_0$. In terms of the inhomogeneous coordinates $t_1 = \frac{x_1}{x_0}$ and $t_2 = \frac{x_2}{x_0}$ on the torus, these are precisely

$$1, t_1, t_2 \tag{1.41}$$

which means the polytope is the convex hull of

$$\{(0,0), (1,0), (0,1)\} \tag{1.42}$$

We thus get a important corollary: The integer points of the polytope associated to a toric variety X_Σ with toric line bundle $\mathcal{O}(\sum_\rho a_\rho D_\rho)$(in which $a_\rho \geq 0$) are

$$\{m \in M | \langle m, v_\rho \geq -a_\rho \rangle\} \tag{1.43}$$

To sum up, the definition above tell us construction from polytopes not only considers the base space, but also gives a specific line bundle of the space. This is a hint that if the property we care about is the full space of a fiber bundle, we need to use polytopes rather than fans.

2 Batyrev's construction

To discuss what Batyrev's theory explains,we need to clarify what a reflective polytope is. We strict our discussion on integral polytopes when we talk about polytopes, but if the concept of polar polytope is introduced

$$\Delta^\circ = \{v \in N_\mathbb{R} : \langle m, v \rangle \geq -1, \forall m \in \Delta\} \subset N_\mathbb{R} \tag{2.1}$$

it is obvious that integral polytope's polar polytope is not always integral, and whose polar is also integral is thus named reflective.

2.1 When is the polytope reflective

One simple fact we can observe without proving is Δ is reflective if and only if Δ° is also reflective.But this is not enough, a more careful examination exposes that we need to introduce some more concepts.

We first recall the concept of canonical bundle, which is a line bundle that associates to the canonical form in the maximum tensor product of cotagent bundle.From the relation of line bundle and divisor, we can define a canonical divisor.A fano space is when the anticanonical bundle is ample(recall ample is when the section gives a embedding into projective space).

One crucial obsevation is Δ is reflective if and only if $\mathbb{P}_\Delta$ is Fano. This proof is not hard through another definition of reflective:

(i)All faces Γ of Δ is supproted by an affine hyperplane of the form $\{m \in M_\mathbb{R} : \langle m, v_\Gamma \rangle = -1\}$ for some $v_\Gamma \in N$

(ii)$Int(\Delta) \cap M = \{0\}$

Fano variety is not hard to check for weighted projective space $\mathbb{P}(q_0, ..., q_n)$,if we let

$$q = \sum_{i=0}^{n} q_i \tag{2.2}$$

then the weighted projective is Fano if an only if $q_i | q$ for all i.We skip the proof.

2.2 how polytope construction associates to Calabi-Yau

It is known that toric variety itself as a Calabi-Yau manifold is not compact, which is not a good news for physicist since compactification needs a compact internal space to reduce those extra dimension. However, polytope construction gives a elegant solution to this problem by considering a natural hypersurface on toric variety by

$$a_0 w^{m_0} + ... + a_k w^{m_k} = 0 \tag{2.3}$$

defines a hypersurface in a n dimensional torus for any given coefficients $a_0, ..., a_k \in \mathbb{C}$,and the closure of this in X_Δ is then a hypersurface. Moreover, if Δ is reflective every such hypersurface is a divisor in the anticanonical class $-K_{X_\Delta}$.

A important fact is all such hypersurface is Calabi-Yau surface when the hypersurface has a dimension bigger than 1. This is proved easily by the language of line bundle and canonical bundle,since

$$K_Y \simeq (K_X \times \mathbb{X}(Y))|Y \tag{2.4}$$

and let

$$K_X \times \mathbb{X}(Y) \simeq \mathbb{O}_X \tag{2.5}$$

then

$$\mathbb{O}_X(Y) \simeq K_X^{-1} \tag{2.6}$$

One important example is the quintic threefold as a hypersurface in the toric variety in $\mathbb{P}^4$,let Δ_n denote the standard simplex $Conv(0, e_1, ..., e_n)$ in $\mathbb{R}^n$,and we take the polytope to be

$$5\Delta_4 - (1,1,1,1) = \{a \in \mathbb{R}^4 | a = 5b - (1,1,1,1) \forall b \in \Delta_4\} \tag{2.7}$$

In other words, the polytope we take is the convex envelope of the vectors

$$\{(-1,-1,-1,-1),(4,-1,-1,-1),(-1,4,-1,-1),(-1,-1,4,-1),(-1,-1,-1,4)\} \tag{2.8}$$

this is a reflective polytope, we can consider its natural defined hypersurface

$$f = \sum_{i=1} a_i t^{m_i} \tag{2.9}$$

where $\Delta \cap M = \{m_1, ..., m_s\}$ and homogenization of the polynomial is

$$F(x_0, x_1, x_2, x_3, x_4) = x_0 x_1 x_2 x_3 x_4 f(x_1/x_0, x_2/x_0, x_3/x_0, x_4/x_0) \tag{2.10}$$

or in a another aspect, we can easily know the fan of its polar polytope

$$\{(-1,-1,-1,-1),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\} \tag{2.11}$$

One important fact and the only thing we should remember in this subsection is that a reflexive polytope itself naturally induces a hypersurface which is Calabi-Yau,this is a rather complicated work to prove with using some algebraic geometry language, and I simply pasted the screen shot of this proof in appendix, which refers to the book "Mirror symmetry and Algebraic geometry".

2.3 the Batyrev's construction

It came quite trivial when all facts are known, naturally reflexive polytopes are connected to Calabi-Yau hypersurfaces and we draw all facts we know in this graph

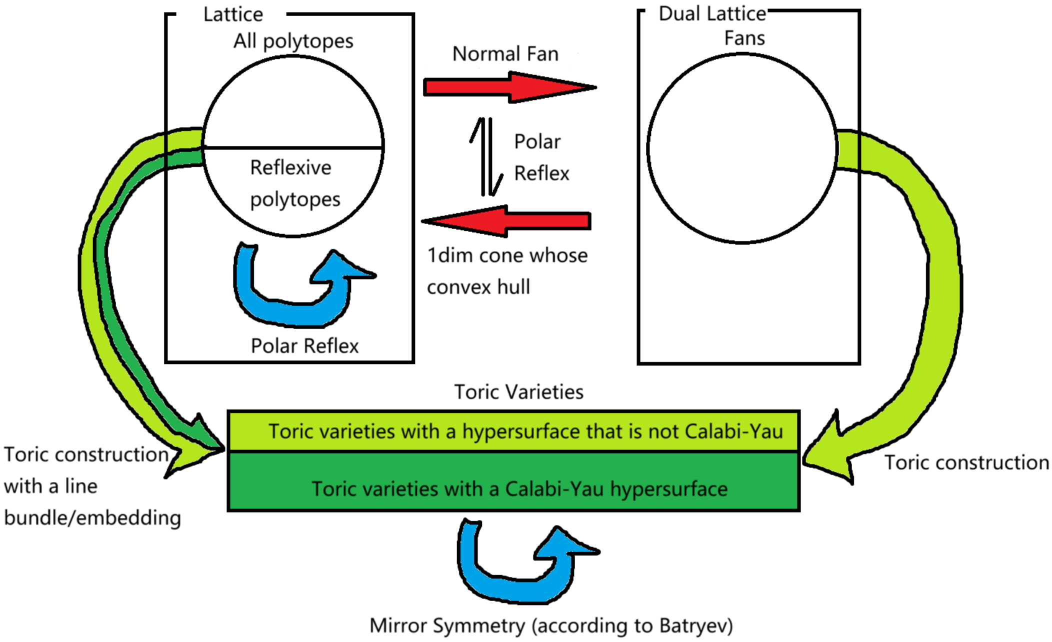

Figure 1.

when two polytopes are polar polytope of each other, Batyrev argues that their natural hypersurface are mirror symmetry pairs ,if V° is the Batyrev Mirror of V, then

$$h^{1,1} = h^{n-2,1}(V^\circ) \quad and \quad h^{n-2,1}(V) = h^{1,1}(V) \tag{2.12}$$

and he also generally suggests

$$h^{p,q}(V) = h^{n-1-p,q}(V^\circ) \tag{2.13}$$

3 the Quintic Threefold

When P.Candelas and other physicists shocked the mathematicians at late 1990 with the calculation of Gromov-Witten invariants to arbitary genus, this simple and paradigmatic model of quintic threefold became renowned.

We have calculated the vertices of the polar polytope as

$$\{(-1,-1,-1,-1),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\} \tag{3.1}$$

and clearly the integral point is

$$\{(-1,-1,-1,-1),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,0,0,0)\} \tag{3.2}$$

with all vertices and the origin,we denote the toric variety generated by this polytope as $\mathbb{P}_{\triangle^\circ}$.We use the construction of the embedding

$$f^\circ : (\mathbb{C}^*)^4 \to \mathbb{P}^5 \tag{3.3}$$

$$[z_1 : z_2 : z_3 : z_4] \to [1 : z_1^{-1} z_2^{-1} z_3^{-1} z_4^{-1} : z_1 : z_2 : z_3 : z_4] \tag{3.4}$$

which is $\mathbb{P}_{\Delta^\circ} \subset \mathbb{P}^5$ defined by

$$y_0^5 = y_1 y_2 y_3 y_4 y_5 \tag{3.5}$$

but this is not the desired embedding, a better construction is as a hypersurface of $\mathbb{P}^4$,we consider a $\mathbb{P}^4$ with homogeneous coordinate as

$$[w_0 : w_1 : w_2 : w_3 : w_4] \tag{3.6}$$

let the group $(\mathbb{Z}_5)^3$ act on $\mathbb{P}^4$ diagonally,as

$$(\mathbb{Z}_5)^3 = \{(w_0, w_1, w_2, w_3, w_4) \in (\mathbb{C}^*)^5 | w_i^5 = 1 \forall i, \prod_{i=0}^{4} w_i = 1\}/G \tag{3.7}$$

and $G = \{(w, w, w, w, w)\}$ is the subgroup of elements that act trivially on $\mathbb{P}^4$. Then there is a map

$$[\mathbb{P}^4/(\mathbb{Z}_5)^3] \to \mathbb{P}^5 \tag{3.8}$$

$$[\hat{x}_1 : \hat{x}_2 : \hat{x}_3 : \hat{x}_4 : \hat{x}_5] \to [\hat{x}_1\hat{x}_2\hat{x}_3\hat{x}_4\hat{x}_5 : \hat{x}_1 : \hat{x}_2 : \hat{x}_3 : \hat{x}_4 : \hat{x}_5] \tag{3.9}$$

this map is an isomorphism onto $\mathbb{P}_{\Delta^\circ} \subset \mathbb{P}^4$ so any of such mirror quintic can be expressed as

$$\{x_1^5 + ... + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0\} \subset [\mathbb{P}^4/(\mathbb{Z}_5)^3] \tag{3.10}$$

and the constant ψ which varies is the modulus.

3.1 how topological invariants are computed

To calculate the topological invariants of a hypersurface of the projective space, we first calculate the projective space.

Using the Whitney sum formula of Chern class,we get

$$c(T_{\mathbb{P}^n}) = \frac{c(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(1+n)})}{\mathcal{O}_{\mathbb{P}^n}} \tag{3.11}$$

assume that $c_1(\mathcal{O}(1)) = h$,then

$$c(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(1+n)}) = \prod_{i=1}^{n+1}(1+h) = (1+h)^{n+1} \tag{3.12}$$

and $c(\mathcal{O}_{\mathbb{P}^n}) = 1$,so

$$c(T_{\mathbb{P}^n}) = (1+h)^{n+1} \tag{3.13}$$

it's clear that

$$c_k(T_{\mathbb{P}^n}) = \mathtt{C}_{n+1}^k h^k \tag{3.14}$$

from the Chern class of the tangent bundle and the relation between tangent bundle and normal bundle, we get

$$c(TX) = \frac{c(T_{\mathbb{P}^n}|_X)}{c(N_X)} \tag{3.15}$$

and

$$N_X \simeq \mathcal{O}_X(d) \tag{3.16}$$

so

$$c(N_X) = 1 + dh \tag{3.17}$$

thus for quintic

$$c(TX) = \frac{(1+h)^5}{1+5h} \tag{3.18}$$

we get

$$c_3(TX) = -40h^3 \tag{3.19}$$

with the normalization

$$\int_X h^3 = deg(X) \int_{\mathbb{P}^3} h^3 = 5 \tag{3.20}$$

then

$$\chi = \int_X c_3(TX) = -200 \tag{3.21}$$

Using the Lefschetz hyperplane theorem, we get that

$$b_0 = 1 \quad b_1 = 0 \quad b_2 = 1 \tag{3.22}$$

and Poincare duality

$$b_6 = 1 \quad b_5 = 0 \quad b_4 = 1 \tag{3.23}$$

so

$$b_3 = 204 \tag{3.24}$$

according to the definition of Euler characteristic.

Then we get the hodge diamond with respect to Calabi-Yau manifold's symmetry on hodge diamonds.

$$h^{2,1} = 101, h^{1,1} = 1 \tag{3.25}$$

A reflexive polytope and Calabi-Yau

PROPOSITION 4.1.3. *If Δ is a reflexive polytope of dimension n, then the general member $\bar{V} \in |-K_{\mathbb{P}_\Delta}|$ is a Calabi-Yau variety of dimension $n-1$. Furthermore, if Σ is a projective subdivision and $X = X_\Sigma$, then:*

(i) The general member $V \in |-K_X|$ is a Calabi-Yau orbifold.

(ii) If Σ is maximal, then the general member $V \in |-K_X|$ is a minimal Calabi-Yau orbifold.

PROOF. We first consider $\bar{V}$. By the definition of Calabi-Yau variety given in Section 1.4, we must show that $\bar{V}$ has canonical singularities, a trivial canonical sheaf, and vanishing cohomology $H^k(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for $0 < k < n-1$.

Since the Fano toric variety $\mathbb{P}_\Delta$ is Gorenstein, it has at most canonical singularities (see [**Batyrev4**]). Then a version of the Bertini theorem guarantees that the general member $\bar{V} \in |-K_{\mathbb{P}_\Delta}|$ also has at most canonical singularities [**Reid1**, Theorem 1.13]. Also, note that

$$\widehat{\Omega}^{n-1}_{\bar{V}} \simeq \widehat{\Omega}^n_{\mathbb{P}_\Delta}(-K_{\mathbb{P}_\Delta}) \otimes \mathcal{O}_{\bar{V}} \simeq \mathcal{O}_{\bar{V}},$$

where the first isomorphism is the adjunction formula (which holds since $\mathbb{P}_\Delta$ is Cohen-Macaulay and $-K_{\mathbb{P}_\Delta}$ is Cartier). The final step is to show that $H^k(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$ for $0 < k < n-1$. Since $\mathcal{O}_{\mathbb{P}_\Delta}(-\bar{V}) \simeq \mathcal{O}_{\mathbb{P}_\Delta}(K_{\mathbb{P}_\Delta}) = \widehat{\Omega}^n_{\mathbb{P}_\Delta}$, we get an exact sequence

$$0 \longrightarrow \widehat{\Omega}^n_{\mathbb{P}_\Delta} \longrightarrow \mathcal{O}_{\mathbb{P}_\Delta} \longrightarrow \mathcal{O}_{\bar{V}} \longrightarrow 0,$$

which gives the long exact sequence

$$\cdots \longrightarrow H^k(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) \longrightarrow H^k(\bar{V}, \mathcal{O}_{\bar{V}}) \longrightarrow H^{k+1}(\mathbb{P}_\Delta, \widehat{\Omega}^n_{\mathbb{P}_\Delta}) \longrightarrow \cdots$$

However, $H^k(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) = 0$ for $k > 0$ and, by Serre-Grothendieck duality, we have $H^{k+1}(\mathbb{P}_\Delta, \widehat{\Omega}^n_{\mathbb{P}_\Delta}) \simeq H^{n-k-1}(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta})^* = 0$ for $k < n-1$. This implies the desired vanishing of $H^k(\bar{V}, \mathcal{O}_{\bar{V}})$, and we conclude that $\bar{V}$ is Calabi-Yau.

Now let Σ be a projective subdivision and set $X = X_\Sigma$. Since $-K_X$ is semi-ample by Lemma 4.1.2, the linear system $|-K_X|$ has no basepoints. Furthermore, X is an orbifold since Σ is simplicial, and then the Bertini theorem (applied to the fixed loci of the local quotients defining X as an orbifold) guarantees that the general member $V \in |-K_X|$ is a suborbifold of X. Everything we did above remains true, and it follows that V is a Calabi-Yau orbifold.

Finally, suppose that Σ is maximal. According to Definition 1.4.1, V is a minimal Calabi-Yau provided it has Gorenstein $\mathbb{Q}$-factorial terminal singularities. Since V is already a Gorenstein orbifold, it automatically has Gorenstein $\mathbb{Q}$-factorial singularities. Hence, we need only show that V has terminal singularities. However, the ambient space X is terminal by Lemma 4.1.2, so we are done by using Bertini as in the proof of [**Reid1**, Theorem 1.13]. □

Figure 2.

B Calculation of polar polytopes

A simple Mathematica program can be used to solve the inequality of polar polytopes (with the help from Deepseek to debug)

Listing 1. Compute polar polytope

```
ClearAll["Global`*"];

```

```
polarPolytopeVertices[vertices_List] :=
Module[{n, allSubsets, polarVertices = {}, subset, A, b,
   candidate,
        k, satisfied},
n = Length[First[vertices]];
If[Length[vertices] != n + 1,
Print["Error:␣The␣number␣of␣vertices␣should␣be␣", n + 1,
",␣but␣got␣", Length[vertices]];
Return[$Failed]];

allSubsets = Subsets[Range[Length[vertices]], {n}];

Do[
subset = allSubsets[[s]];
A = vertices[[subset]];
If[Det[A] != 0,
b = -ConstantArray[1, n];
candidate = LinearSolve[A, b];
satisfied = True;
For[k = 1, k <= Length[vertices], k++,
If[vertices[[k]].candidate < -1, satisfied = False; Break
   [];]
];
If[satisfied, AppendTo[polarVertices, candidate]]
];
, {s, 1, Length[allSubsets]}];

DeleteDuplicates[polarVertices]
]
```

For example

```
verticesQuintic = {{-1, -1, -1, -1}, {4, -1, -1, -1}, {-1,
   4, -1, -1}, {-1, -1, 4, -1}, {-1, -1, -1, 4}};
polarPolytopeVertices[verticesQuintic]
```

and we get

```
{{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0},
   {-1, -1, -1, -1}}
```

C useful relations of Chern class and topological invariants

C.1 Lefschetz hyperplane theorem

With X is a n-dimensional smooth connected complex projective variety embedded in $\mathbb{P}^n$,H is a hypersurface of $\mathbb{P}^n$, $Y = X \cap H$. Lefschetz said that:Considering $i : Y \to X$

1.$i_* : H_k(Y,\mathbb{Z}) \simeq H_k(X,\mathbb{Z})$ for any $k < n-1$,and i_* is a surjection when $k = n-1$

2.$i_* : \pi_k(Y,\mathbb{Z}) \simeq \pi_k(X,\mathbb{Z})$ for any $k < n-1$

C.2 Tangent and normal bundle of hypersurface

As $X \subset Y$ is a smooth hypersurface, its normal bundle

$$N_X = \mathcal{O}_Y(X)|_X \tag{C.1}$$

is limited by the line bundle of X,and for $Y = \mathbb{P}^n$ and X is a hypersurface of degree d,

$$N_X \simeq \mathcal{O}_X(d) \tag{C.2}$$

we also have short exact sequence

$$0 \to TX \to TY|_X \to N_X \to 0 \tag{C.3}$$

which enables

$$c(TX) = \frac{c(TY|_X)}{c(N_X)} \tag{C.4}$$

C.3 Tangent bundle's properties of projective space

We assume that the homogeneous coordinate of $\mathbb{P}^n$ is $[x_0 : x_1 : ... : x_n]$, and a tangent vector is the generator of the coordinate perturbation

$$[x_0 + \epsilon s_0 : ... : x_n + \epsilon s_n] \tag{C.5}$$

if the perturbation do not changes the coordinate respect to the equivalent relation, we have

$$s_i = \lambda x_i \tag{C.6}$$

we now construct two maps,the first one is from a trivial line bundle to the direct sums of line bundles

$$f : \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \tag{C.7}$$

$$1 \to [x_0 : x_1 : ... : x_n] \tag{C.8}$$

and the second one is from the direct sums of line bundles to the tangent bundle,

$$g : \mathcal{O}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \tag{C.9}$$

$$g : [s_0 : s_1 : ... : s_n] \to \sum_{i=0}^{n} s_i \frac{\partial}{\partial x_i} \tag{C.10}$$

and we can confirm that the kernel of the second map is the image of the first, to say

$$\sum x_i \frac{\partial}{\partial x_i} = \sum_{k \neq j} x_k \frac{\partial y_k}{\partial x_k} \frac{\partial}{\partial y_k} + x_j(-\sum_{k \neq j} \frac{x_k}{x_j^2} \frac{\partial}{\partial y_k}) = 0 \tag{C.11}$$

in tangent vectors.Thus we get

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes(n+1)} \to T_{\mathbb{P}^n} \to 0 \tag{C.12}$$

References

[1] S. Hosono, A. Klemm and S. Theisen, *Lectures on Mirror Symmetry*, arxiv:9403096.

[2] Emily Clader and Yongbin Ruan, *Mirror Symmetry Constructions* arxiv:1412.1268

[3] David A.Cox and Sheldon Katz, *Mirror Symmetry and Algebraic Geometry*, 1991.